BASIC SEQUENCES AND NORMING SUBSPACES IN NON-QUASI-REFLEXIVE BANACH SPACES

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ABSTRACT

A Banach space X is non-quasi-reflexive (i.e. dim $X^{**}/X = \infty$) if and only if it contains a basic sequence spanning a non-quasi-reflexive subspace. In fact, this basic sequence can be chosen to be non-k-boundedly complete for all k. A basic sequence which is non-k-shrinking for all k exists in X if and only if X^* contains a norming subspace of infinite codimension. This need not occur even if X is non-quasi-reflexive. Every norming subspace of X^* has finite codimension if and only if for every norming M in X^*, every M-closed Y in X, $M \cap Y^{\top}$ is norming over X/Y. This solves a problem due to Schäffer [19].

1. Introduction and notation

J. J. Schäffer [19] asked *if M is a norming subspace of X^* and Y is an M-closed subspace of X, then is $M \cap Y^{\perp}$ a norming subspace* of $(X/Y)^*$? (Here $Y^{\perp}$ is identified with $(X/Y)^*$ in the canonical way. Relevant definitions appear below.) In Section 2 we give a counter-example, and in Section 4 we show that such a counter example can be constructed for X if and only if X^* contains an infinite codimensional norming subspace.

Section 2 consists of the above mentioned example and two other examples concerning norming subspaces. Example 1 shows that non-quasi-reflexivity of X does not yield the existence in X^* of an infinite codimensional norming subspace. Example 3 gives an X such that X^* contains a total subspace which is not norming over any subspace of X. (The examples in [8] and [14] of total, non-norming subspaces of X^* are constructed in such a way that they norm some subspace of X.)

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In order to give the complete solution to the Schäffer problem, it was necessary to prove new existence theorems for basic sequences in Banach spaces. We prove in Section 3 that (a) If X^* contains an infinite codimensional norming subspace, then X contains a basic sequence (x_n) which is not k-shrinking for any k; i.e., the functionals on $[x_n]$ biorthogonal to (x_n) span an infinite codimensional subspace of $[x_n]^*$, and (b) every non-quasi-reflexive space contains a basic sequence, (x_n), which is not k-boundedly complete for any k; in fact, $\| x_n \| = 1$ and

$$\left(\left\| \sum_{k=i}^{p} x_{n_k+i} \right\| \right)_{i=0,\, p=i}^{\infty \quad \infty}$$

$(n_k = 2^{-1}k(k+1))$ is bounded. (Example 2 shows that the hypothesis on X in (a) cannot be weakened to "X is non-quasi-reflexive"). The proof of (a) uses an extension of Pelczynski's technique [15] for constructing nonshrinking basic sequences in nonreflexive spaces. The proof of (b) is rather complicated and not at all similar to the usual method for constructing non-boundedly complete basic sequences in nonreflexive spaces.

Of course, an immediate application of (b) is that every non-quasi-reflexive space contains a non-quasi-reflexive subspace with basis. Other applications are that if X is not quasi-reflexive, then $X \supset Y$ such that Y and X/Y are not quasi-reflexive; and if X^* contains an infinite codimensional norming subspace, then X has a subspace Y such that X^* and $(X/Y)^*$ both contain infinite codimensional norming subspaces. This last application is used in solving the Schäffer problem.

We now explain the terminology used. X, Y, Z, etc. refer to infinite dimensional Banach spaces. Subspaces are assumed closed. A space X is quasi-reflexive [3] provided that X has finite codimension in X^{**}. (We always assume X is canonically embedded in X^{**}.) A subspace Z, of X^* is norming over a subspace Y of X provided that there is a constant λ such that $\| y \| \leq \lambda \sup \{ | z(y) | : z \in Z, \| z \| \leq 1 \}$ for each $y \in Y$. Then we say that Z is λ-norming over Y. When $Y = X$, we say Z is norming and call the smallest such constant λ the norming constant of Z. Every norming subspace of X^* has finite codimension in X^* when X is quasi-reflexive, but Example 2 shows that the converse is false.

For $A \subset X$, $A^\perp$ is the annihilator of A in X^*. For $A \subset X^*$, $A_\perp$ is the annihilator of A in X and $\tilde{A}$ is the weak* closure of A in X^*. A subspace M of X^* is total over X provided $M_\perp = \{0\}$. Note that, given M a subspace of X^* and Y a subspace of X, Y is M-closed (i.e., Y is closed in X when X is given the weak topology from M) if and only if $M \cap Y^\perp$ is total over X/Y. Norming subspaces are total, but every total subspace of X^* is norming only if X is quasi-reflexive, (cf. [5]).

We use standard facts about basic sequences (cf. [21]). Suppose (x_n) is a basic sequence with biorthogonal functionals (x_n^*) in $[x_n]^*$ and k is a non-negative integer. Then (x_n) is k-shrinking (resp., k-boundedly-complete) (cf. [20]) provided $[x_n^*]$ has codimension k in $[x_n]^*$ (resp., the image of $[x_n]$ under the canonical embedding of $[x_n]$ into $[x_n^*]^*$ has codimension k in $[x_n^*]^*$). The partial sum operators are defined by $S_n(u) = \sum_{i=1}^{n} x_i^*(u)x_i$, and the basis constant is simply $\sup \| S_n \|$. A sequence $(f_n) \subset X^*$ is weak*-basic if there is a sequence $(x_n) \subset X$ biorthogonal to (f_n) such that $u \in \widetilde{[f_n]}$ implies $\sum u(x_n)f_n$ converges weak* to u.

We would like to thank Professor Schäffer for calling our attention to his problem.

2. Some examples

The Schäffer problem has an affirmative answer when M is close to being minimal or maximal. Let M be a norming subspace of X^* and Y an M-closed subspace of X. Suppose first that the natural map $T: X \to M^*$ defined by $(Tx)m = m(x)$ has finite codimensional range in M^*. Then the natural map from X/Y into $(M \cap Y^\perp)^*$ is one-to-one and has a finite codimensional (hence closed) range. Therefore this map is an isomorphism, whence $M \cap Y^\perp$ is norming over X/Y. Secondly, suppose that M has finite codimension in X^*. Then $M \cap Y^\perp$ has finite codimension in $Y^\perp$ and is total over X/Y, hence is norming over X/Y by [8].

In particular, Schäffer's question has an affirmative answer when considering spaces X such that every norming subspace of X^* has finite codimension in X^*. Of course this happens when X is quasi-reflexive. Surprisingly, this can also occur when X is not quasi-reflexive, as illustrated by the following example, which was discovered in conversation with D. W. Dean.

Example 1. Let Z be the James-Lindenstrauss space with $Z^{**}/Z = c_0$(cf. [13]). If M is a norming subspace of Z^*, then $Z \cap M^\perp = \{0\}$ and $Z + M^\perp$ is closed in Z^{**} (cf. [8]), so the quotient map $Q: Z^{**} \to c_0$ has $Q_{|M^\perp}$ an isomorphism. Therefore $M^\perp$ is both a subspace of the separable conjugate space Z^{**} and is isomorphic to a subspace of c_0. Thus by [2] and [17], $M^\perp$ is finite dimensional, proving that M has finite dimension in Z^*.

This example has another curious property. By Lemma 3 in Section IV below, if Y is a subspace of Z, then every norming subspace of Y^* has finite codimension in Y^*. In particular, if (z_n) is a basic sequence in Z with biorthogonal functionals (z_n^*) in $[z_n]^*$, then $[z_n^*]$ has finite codimension in $[z_n]^*$. That is, every basic sequence in Z is k-shrinking for some k. On the other hand, since Z has a shrinking basis (cf. [13]), it follows from [7] that, for each k, Z has a basis which is k-shrinking.

The second example gives a negative solution to the Schäffer problem. The complete solution to Schäffer's question given in Section 4 was motivated by this construction.

EXAMPLE 2. Let E be non reflexive and let $X = (\sum E)_{c_0} = \{(e_j) \mid e_j \in E \ \forall j \ e_j \to 0, \ \|(e_j)\| = \sup \|e_j\| < \infty\}$. Since E is nonreflexive, E^* can be written as $H \oplus [\phi]$ with H norming. If we let ϕ_j denote ϕ in the j-th copy of E^* in $(\sum E^*)_{l_1} = (\sum E)_{c_0}^*$, then $[\phi_j]$ is a weak* closed subspace of X^*: Let $\sum a_{nj}\phi_j \overset{w^*}{\to} (f_j)$. If π_i is the i-th coordinate projection on X, then $\pi_i^*(\sum a_{nj}\phi_j) = a_{ni}\phi_i \to f_i$, so that $f_i = a_i\phi_i$ for some a_i. Since $\sum \|f_i\| = \|(f_i)\|$, it follows that $(a_i) \in l_1$ so that $(a_i\phi_i) = (f_i) \in [\phi_j]$. With the assumption that $\|\phi\| = 1$, we can choose $z \in E$ with $\|z\| < 2$ and $\phi(z) = 1$. If we define $P : E \to [z]$ by $P(e) = \phi(e)z$, P is a projection with $\|P\| < 2$. Letting P_j denote this projection in the j-th copy of E in X, it is easy to see that $(\sum P_j)((e_j)) = (P_je_j)$ is a projection of X onto $[z_j]$ which in turn is isomorphic to c_0 in the natural way. If we now let $Y = [\phi_j]_\perp$, it follows that X/Y is isomorphic to c_0, so that $Y^\perp = [\phi_j]$ contains a total, non-norming subspace M_1[14]. Now let H_j denote H in the j-th copy of E^* in X^*. It is easy to see that $M_2 = [H_j]$ is a norming subspace of X^*.

Finally, let $M = M_1 + M_2$. Since $M \supset M_2$, M is closed and norming. Also, $M \cap Y^\perp = M_1$, so that $M \cap Y^+$ is total but non-norming over X/Y. The totality of $M \cap Y^\perp$ guarantees that Y is M-closed.

Using ideas of Bessaga and Pelczynski [1], J. C. Daneman proved that every subspace of l_1 is norming over some subspace of c_0. Our last example shows that there exists a Y and a total subspace of Y^* which is not norming over any subspace of Y.

EXAMPLE 3. Let Y be the space of James-Lindenstrauss [13] satisfying $Y^{**} = Y \oplus l_1$. Y is just the conjugate to the space X of Example 1. Y was constructed so that there is a quotient mapping $Q : Y^* \xrightarrow{\text{onto}} c_0$ with $Q^*\delta_n = (0, \delta_n)$ (where (δ_n) is the usual basis for l_1). Thus, $(0, \delta_n) \overset{w^*}{\to} 0$. Y has a normalized shrinking basis (x_n), and of course, $x_n \overset{w}{\to} 0$.

Let p be a mapping of the integers onto themselves so that $p^{-1}(m)$ is infinite for each m. Let $Z = [(x_{p(n)}, \delta_n)_{n=1}^{\infty}]$. Then $(x_{p(n)}, \delta_n)$ is equivalent to the usual basis for l_1 because

$$\sum_{i=1}^{n} |\alpha_i| = \left\| \sum_{i=1}^{n} \alpha_i(0, \delta_i) \right\| \leqq \left\| \sum_{i=1}^{n} \alpha_i(x_{p(i)}, \delta_i) \right\|$$

for all choices of scalars (α_i).

We claim that Z is norming over no infinite dimensional subspace of Y^*. For suppose W is a subspace of Y^* and Z is norming over W. Then, since $(x_{p(n)}, \delta_n)$ is equivalent to the usual basis for l_1, there is a constant λ such that $\| w \| \leqq \lambda \sup_n (x_{p(n)}, \delta_n) w$ for $w \in W$. Thus, letting $K = \overline{(x_{p(n)}, \delta_n)}$, we have that the map $T\colon W \to C(K)$ defined by $Tw(k) = k(w)$ is an isomorphism into. But it is easily seen from the facts that $x_n \xrightarrow{w} 0$ and $(0, \delta_n) \xrightarrow{w^*} 0$ that $K = (x_{p(n)}, \delta_n) \cup (x_n, 0) \cup \{(0,0)\}$, so that K is countable. Hence by [17], TW and also W contain a subspace isomorphic to c_0, which by [2] contradicts the separability of Y^*.

However, Z is total. For suppose $y^* \in Z_\perp$ and $n \in p^{-1}(m)$. Then $0 = (x_{p(n)}, \delta_n)(y^*) = y^*(x_m) + (0,\delta_n)y^*$. Letting $n \to \infty$ through $p^{-1}(m)$, we have $y^*(x_m) = 0$. Since m is arbitrary, $y^* = 0$.

3. Basic sequences in non-quasi-reflexive spaces

It is relatively easy (proof of Proposition 1) to find a separable non-quasi-reflexive subspace of an arbitrary non-quasi-reflexive space. Here we show that in fact, the separable subspace may be chosen to have a basis.

In the remainder of the paper we use the notation $n_k = 2^{-1}k(k+1)$. Note also that we index sequences from 0 to ∞.

THEOREM 1. *If Y^* contains a norming subspace of infinite codimension, then Y contains a basic sequence which is not k-shrinking for any k. In particular, if X is non-quasi-reflexive then X^* contains a basic sequence which is not k-shrinking for any k.*

PROOF. Let $N \subset Y^*$ be norming with Y^*/N infinite dimensional. Then Y^*/N admits a biorthogonal system $(u_n; u_n^*)$ satisfying $\| u_n \| = \| u_n^* \| = 1$, [6]. Thus, there is a biorthogonal system $(z_n; z_n^*)$ with (z_n^*) unit vectors in $N^\perp$, $(z_n) \subset Y^*$ and $\| z_n \| \leqq 1 + 1/(n+1)$. Let λ be the norming constant of N and let $\varepsilon_i \to 0$ with $\varepsilon_i > 0$ for $i = 0, 1, \cdots$. We choose (x_n) and finite dimensional subspaces $F_0 \subset F_1 \subset \cdots$ of N to satisfy

1) F_i is $\lambda(1+\varepsilon_i)$-norming over $[x_k | 0 \leqq k \leqq i]$, for $i = 0, 1, \cdots$,
2) $x_{i+1} \in (F_i)_\perp$ for $i = 0, 1, \cdots$,
3) $z_k(x_{n_p+j}) = \delta_{kj}$ for $k = 0, 1 \cdots, p$; $j = 0, 1, \cdots, p$, $p = 0, 1, \cdots$, and
4) $\| x_i \| \leqq 1 + 1/(i+1)$ for $i = 0, 1, \cdots$.

To see that these conditions yield the conclusion, notice that (1) and (2) are the standard constructive conditions which guarantee that (x_i) is basic with the n-th partial sum operator S_n satisfying $\| S_n \| \leqq \lambda(1+\varepsilon_n)$. Since $(x_{n_k+k}; z_k)$ is a biorthogonal system, condition (3) implies the linear independence of $(z_{k|[x_n]})$. Finally, no z_k can be in the closed span of the coefficient functionals since $z_k(x_j) \not\to 0$ as $j \to \infty$ and $\| x_i \| \leqq 2$ for all i. Thus, (x_j) is non-k-shrinking for every k. The final assertion follows because X is a norming subspace of infinite codimension in X^{**} if and only if X is non-quasi-reflexive.

We now select the desired sequences inductively. By Helly's theorem, pick x_0 so that $\| x_0 \| < 2$ and $z_0(x_0) = 1$. Choose F_0 to satisfy (1). Now suppose we have already chosen $(x_0, \cdots, x_{n_p+j})$ and $F_0 \subset \cdots \subset F_{n_p+j}$. Suppose $j < p$. By Helly's theorem, we can pick x_{n_p+j+1} with $\|x_{n_p+j+1}\| < 1 + 1/(n_{p+j+2})$ and such that x_{n_p+j+1} agrees with z^*_{j+1} on $[F_{n_p+j}, z_0, \cdots, z_p]$. This x_{n_p+j+1} satisfies (2), (3), and (4). In case $j = p$, $n_p + j + 1 = n_{p+1}$, and the only change from the case $j < p$ is that one wants $x_{n_{p+1}}$ to agree with z_0^* on $[F_{n_p+j}, z_0, \cdots, z_{p+1}]$. In each case, simply choose $F_{n_p+j+1} \supset F_{n_p+j}$ to satisfy (1). Q.E.D.

REMARK 1. If S_n denote the n-th partial sum operator on $[x_n]$ and if $\bar{z}_k = z_{k|[x_n]}$, set $f_k = (I - S_{n_k})^* \bar{z}_k$ for each k. It is immediate that $f_k(x_{n_p+j}) = \delta_{kj}$ for all p, k, and $j = 0, 1, \cdots, p$.

REMARK 2. If T is an isomorphism of Y into X^* with X separable, and if T^*X has infinite codimension in Y^*, then the basic sequence (x_n) in Y may be chosen so that (Tx_n) is w^*-basic. In this case, let $N = T^*X$ and in choosing the F_i's be sure also that $[F_i] = T^*X$. An easier version of the proof of Th. 3.1 of [10] shows that (Tx_i) is weak* basic.

Applying this remark to $Y = X^*$ where X is separable and non-quasi-reflexive, we have that X has a quotient space with a basis which fails to be k-boundedly-complete for every k (see, e.g. [20]).

The next corollary is used in Section 4 in providing the complete solution to the Schäffer problem of Section 1.

COROLLARY 1. *If Y^* contains a norming subspace of infinite codimension,* *then $Y \supset Z$ such that $(Y/Z)^*$ and Z^* both contain norming subspaces of infinite codimension.*

PROOF. By Theorem 1 and Remark 1, there is a bounded basic sequence $(y_n) \subset Y$ and a bounded sequence $(f_n) \subset Y^*$ such that $f_k(y_{n_p+j}) = \delta_{kj}$ for all p, k and for $j = 0, \cdots, p$. Let $Z = [y_{n_p+j} | 0 \leqq p < \infty, j \text{ even}]$. Then, the natural basis for Z is non-k-shrinking for all k, so its coeffient functionals span a norming subspace of infinite codimension in Z^*. If Q is the quotient map of Y onto Y/Z, then $(Q(y_{n_p+j}) | j \text{ is odd})$ is a bounded basic sequence (cf. [21, p. 26, Proposition 4. 1]) which is non-k-shrinking for all k (since $(f_k | k \text{ is odd})$ is still a linearly independent subset of $Z^\perp = (Y/Z)^*$). Thus, as above, $[Q(y_{n_p+j}) | j \text{ odd}] \subset Y/Z$ has desired property, so Y/Z does also by Lemma 3 in Section 4. Q.E.D.

The remainder of this section is devoted to the construction, in a non-quasi-reflexive space, of a non-quasi-reflexive subspace with basis. This lemma is a technical device needed in the construction:

LEMMA 1. *Suppose F is finite dimensional in X^*, $(y_n) \subset X^*$ is weak* null and basic with $\| y_n \| = 1$ for all n, and $(I_i)_{i=0}^{\infty}$ is a partition of the natural numbers into pairwise disjoint infinite sets. Then for each $\varepsilon > 0$, there exist infinite sets $I'_j \subset I_j$; $j = 0, 1, \cdots$, such that the natural projection onto F from $F \oplus [y_n | n \in \cup I'_j]$ has norm $\leqq 1 + \varepsilon$.*

PROOF. Let U be a finite dimensional subspace of X which is $(1 + \varepsilon/2)$-norming over F. Let K denote the basis constant for (y_n) and let $\varepsilon' = \varepsilon/4K$. For $i = n_p + j$, choose k_i in I_j so that

$$\sup_{\substack{\|u\| \leqq 1 \\ u \in U}} |y_{k_i}(u)| < \frac{\varepsilon'}{2^{i+1}}.$$

Then, for any sequence (a_j) of scalars,

$$\sup_{\substack{\|u\| \leqq 1 \\ u \in U}} |(\Sigma a_i y_{k_i})(u)| \leqq \varepsilon' \{\Sigma |a_i| 2^{-i-1}\}.$$

However, $|a_i| \leqq 2K \| \Sigma a_i y_{k_i} \|$, so we have

$$\sup_{\substack{\|u\| \leqq 1 \\ u \in U}} |(\Sigma a_i y_{k_i})(u)| < \frac{\varepsilon}{2} \| \Sigma a_i y_{k_i} \|.$$

Thus, setting $I'_j = \{k_i | i = n_p + j \text{ for some } p\}$ gives the desired result. (It should should be noted that with a little more work, the a priori assumption that (y_n) is basic may be dropped.) Q.E.D.

The next proposition gives a finite dimensional decomposition of a subspace of X which has the desired properties. The technique is a modification of the Mazur-Day-Gelbaum technique for selecting basic sequences. The primary modification lies in the norming of a finite dimensional subspace of the form $X_0 \oplus X_1 \oplus \cdots \oplus X_n$ at each stage before X_n has been specifically selected.

PROPOSITION 1. *Suppose X is non-quasi-reflexive. Then there is a bounded biorthogonal system $(x_n; h_n)$ with $(x_n) \subset X$ such that $(\Sigma^k_{p=j} x_{n_p+j} \mid 0 \leqq j \leqq k < \infty)$ is bounded, and for x in $[x_n]$,*

$$x = \lim_{p\to\infty} \sum_{n=0}^{n_n+p} h_n(x)x_n .$$

PROOF. Pelczynski [16] showed that for each k, X contains a basic sequence $(u_n^{(k)})$ which is non-l-shrinking for $0 \leqq l \leqq k$. The subspace $[u_n^{(k)} \mid 0 \leqq k,\, n < \infty]$ is therefore separable and non-quasi-reflexive. Thus, with no loss of generality, we assume that X is separable.

By Theorem 1 and Remark 1, there is a bounded basic sequence $(y_n) \subset X^*$, a bounded sequence $(f_n) \subset X^{**}$ and a partition (I_n) of the integers into pairwise disjoint infinite subsets such that

$$f_k(y_n) = \begin{cases} 1 & \text{if} \quad n \in I_k \\ 0 & \text{if} \quad n \notin I_k. \end{cases}$$

Further, we assume by Remark 2 following Theorem 1 that (y_n) is w^*-basic, and hence w^*-null. Let $\lambda > \|f_n\|$ for all n and choose $0 < \varepsilon_i < 1$ for all i with $\varepsilon_i \to 0$.

The main difficulty in the proof is to guarantee that the natural projections of $[x_i]$ onto $[x_0, \cdots, x_{n_p+p}]$ are bounded independently of p. To do this we will inductively define the x_i''s in blocks $(x_{n_p}, \cdots, x_{n_p+p})$ and define finite dimensional subspaces $G_0 \subset G_1 \subset \cdots$ of X^* so that G_p norms $[x_0, \cdots, x_{n_p+p}]$ and $[x_{n_{p+1}}, \cdots] \subset (G_p)_\perp$. It is necessary to replace the f_i's by related functionals, f_i'. At the p-th step of the induction, we will make sure that $f_j'(g) = g(\Sigma^p_{i=j} x_{n_i+j})$ for $g \in G_p$, $1 \leqq j \leqq p$; and $f'_{p+1} \in (G_p)$. Then we pick $G_{p+1} \supset G_p$ so that G_{p+1} norms $Z = [x_0, \cdots x_{n_p+p}, f_1', \cdots, f'_{p+1}]$. Now we use local reflexivity to reflect Z through G_{p+1}; x_j goes to x_j and f_j' goes to $\Sigma^{p+1}_{i=j} x_{n_i+j}$ (this defines $(x_{n_{p+1}} \cdots, x_{n_{p+1}+p+1})$). Thus, $(x_{n_{p+1}}, \cdots, \cdots, x_{n_{p+1}+p+1}) \subset (G_p)_\perp$. Finally, local reflexivity will guarantee that G_{p+1} norms $[x_0, \cdots, x_{n_{p+1}+p+1}]$ and $f_j(g') = g(\Sigma^{p+1}_{i+j} x_{n_i+j})$ for $g \in G_{p+1}$, $1 \leqq j \leqq p+1$.

For the first step, let G_0 be a finite dimensional subspace of X^* which $(1+\varepsilon_0)$-norms $[f_0]$. By Helly's theorem, we pick x_0 in X with $\|x_0\| \leqq \lambda$ such that $g(x_0) = f_0(g)$ for g in G_0. For convenience of notation later, we now rename $f_0 = f_0'$.

Suppose $(x_k \mid 0 \leqq k \leqq n_p + p)$, $(f_i' \mid 0 \leqq i \leqq p)$, and finite dimensional subspaces $G_0 \subset G_1 \subset \cdots \subset G_p$ of X^* have been chosen to satisfy the following conditions:

1) For each j, $0 \leqq j \leqq p$ there are functionals $\phi_0, \cdots, \phi_j$ on $[x_{n_j+i} \mid 0 \leqq i \leqq j]$ of norm $\leqq 2$ with the system $(x_{n_j+i}, \phi_i \mid 0 \leqq i \leqq j)$ biorthogonal.

2) G_i is $(1+\varepsilon_i)^2$ norming over $[x_k \mid 0 \leqq k \leqq n_i + i]$ for $0 \leqq i \leqq p$.

3) $(x_{n_{i+1}+j} \mid 0 \leqq j \leqq i+1) \subset (G_i)_\perp$ for $0 \leqq i \leqq p$.

4) $(\| \sum_{i=j}^{k} x_{n_i+j} \| \mid j \leqq k \leqq p)$ is bounded by 6λ for $0 \leqq j \leqq p$.

5) $g(\sum_{i=j}^{p} x_{n_i+j}) = f_j'(g)$ for $g \in G_p$, $0 \leqq j \leqq p$.

6) $\|f_i'\| < 3\lambda$ for $0 \leqq i \leqq p$.

7) There exist infinite sets $I_k' \subset I_k$, $k = 0, 1, \cdots$, so that for each i, $0 \leqq i \leqq p$, f_i' agrees with f_i on $[y_n \mid n \in \cup I_k']$.

Let (1′),⋯,(7′) be the statements above for $p+1$. By Lemma 1, pick infinite sets $I_k'' \subset I_k'$ for all k so that the natural projection onto G_p from $G_p \oplus [y_n \mid n \in \cup I_k'']$ has norm $\leqq 2$. Hence, there exists f_{p+1}' in X^{**} with $\|f_{p+1}'\| < 3\lambda$ so that $f_{p+1}'(g) = 0$ for $g \in G_p$, and such that f_{p+1}' agrees with f_{p+1} on $[y_n \mid n \in \cup I_k'']$. This satisfies (6′) and (7′).

Since $y_n \xrightarrow{w^*} 0$, and each I_k'' is infinite, there exist, for $0 \leqq i \leqq p+1$, $q_i \in I''$ so that

$$\sum_{k=0}^{n\ +p} | y_{q_i}(x_k) | < 1/4p.$$

Now select a finite dimensional subspace G_{p+1} of X^*, $G_{p+1} \supset G_p \cup (y_{q_i} \mid 0 \leqq i \leqq p+1)$ such that G_{p+1} is $(1+\varepsilon_{p+1})$-norming over $F = [(x_k \mid 0 \leqq k \leqq n_p + p) \cup (f_i' \mid 0 \leqq i \leqq p+1)] \subset X^{**}$. By the principle of local reflexivity [11], there is an operator $T: F \to X$ such that T is the identity on $[x_k \mid 0 \leqq k \leqq n_p + p]$, T is a $(1+\varepsilon_{p+1})$-isometry and $g(Tf) = f(g)$ for $f \in F$, $g \in G_{p+1}$. Define $x_{n_{p+1}}, \cdots, x_{n_{p+1}+p+1}$ by $x_{n_{p+1}+j} = Tf_j' - \sum_{i=j}^{p} x_{n_i+j}$ for $0 \leqq j \leqq p+1$. Thus $Tf_j' = \sum_{i=j}^{p+1} x_{n_i+j}$ for $0 \leqq j \leqq p+1$, so that (4′) and (5′) hold. Since $G_p \subset G_{p+1}$ and $f_{p+1}' \in G_p^\perp$, we have from (5) that (3′) holds. Since G_{p+1} is $(1+\varepsilon_{p+1})$-norming over F, local reflexivity guarantees that G_{p+1} is $(1+\varepsilon_{p+1})^2$-

norming over $[x_n \,|\, 0 \leqq k \leqq n_{p+1} + p + 1]$ so that (2′) holds. Now, for $0 \leqq i \leqq p+1$, $0 \leqq j \leqq p+1$, one again has from local reflexivity that $y_{q_i}(x_{n_{p+1}+j}) = f'_j(y_{q_i}) - \Sigma^p_{k=j}\, y_{q_i}(x_{n_k+j})$, so that $y_{q_i}(x_{n_{p+1}+i}) \geqq 1 - 1/4p \geqq 3/4$ and $|y_{q_i}(x_{n_{p+1}+j})| < 1/4p$ when $i \neq j$. It is easy to compute that the functionals defined by $\phi_i(x_{n_p+j}) = \delta_{ij}$, $i,j = 0,1,\cdots,p+1$, satisfy $\| \phi_i \| \leqq 2$. This completes the construction of (x_n).

From (3) we see that, since $(G_p)_\perp \supset [x_k | k \geqq n_{p+1}]$, and by (2), the natural projection S_p of $[x_n]$ onto $[x_k \,|\, 0 \leqq k \leqq n_p + p]$ has norm $\leqq (1+\varepsilon_p)^2$. Thus, (S_p) determines a finite dimensional decomposition of $[x_n]$, since (if (h_n) are the functionals in $[x_n]^*$ biorthogonal to (x_n)) then

$$S_p(x) = \sum_{k=0}^{n_p+p} h_k(x)x_k.$$

The assertion of the boundness of $(\| \Sigma^k_{p=j}\, x_{n_p+j} \|)$ is guaranteed by (4). Q.E.D.

It should be noted that if, at each stage of the construction above, the collection $(f'_0 - \Sigma^p_{k=0}\, x_{n_k}, f'_1 - \Sigma^p_{k=1}\, x_{n_k+1}, \cdots, f'_p)$ were basic with constant independent of p, then local reflexivity would guarantee the same for $(x_{n_{p+1}}, x_{n_{p+1}+1}, \cdots, x_{n_{p+1}+p+1})$. If we then set $P_{n_r+j} = S_{n_{r-1}} + \sigma_j^{(r)}(I - S_{n_{r-1}})S_{n_r}$ where

$$\sigma_j^{(r)} : [x_{n_r}, \cdots, x_{n_r+r}] \to [x_{n_r}, \cdots, x_{n_r+j}]$$

is the natural projection, we would have $(\|P_{n_r+j}\| \,|\, 0 \leqq j \leqq r < \infty)$ bounded, so that (x_n) would already be a basis for its span. The boundedness of $(\| \Sigma^r_{k=j}\, x_{n_k+j} \|)$ would guarantee that it is non-k-boundedly complete for all k as desired. The rest of this section is devoted to the realization of this situation.

LEMMA 2. *Suppose that (y_n) is a bounded sequence in X with $\inf_{n \neq m} \| y_n - y_m \| = \delta > 0$. Given $\varepsilon > 0$, there exist integers $m_1 < m_2 \cdots$ such that $(y_{2m_i} - y_{2m_i+1})$ is basic with basis constant $\leqq 1 + \varepsilon$.*

PROOF. Let Z be a seperable subspace of X^* with norming constant 1. By passing to a subsequence of (y_n), we may assume that $\lim_i z(y_i)$ exists for each $z \in Z$. Thus $z(y_{2i} - y_{2i+1}) \to 0$ for $z \in Z$ and $\| y_{2i} - y_{2i+1} \| \geqq \delta > 0$. Hence $(y_{2i} - y_{2i+1})$ has a subsequence with the desired properties (cf. [12]). Q.E.D.

THEOREM 2. *Suppose X is not quasi-reflexive. Then X contains a basic sequence (z_n) which is not k-boundedly complete for any k. In fact, (z_n) is bounded away from 0 and $(\| \Sigma^k_{i=j}\, z_{n_i+j} \|)^{\infty}_{j=0}{}^{\infty}_{k=j}$ is bounded.*

PROOF. By Proposition 1, there exists a bounded biorthogonal sequence $(x_n;f_n)$ in X with $\|f_n\| = 1$, $((\| \sum_{i=j}^{k} x_{n_i+j} \|)_{j=0}^{\infty})_{k=j}^{\infty}$ bounded by, say λ, and for each $x \in [x_n]$, $x = \lim_{p\to 0} \sum_{k=0}^{np+p} f_k(x)x_k$. We may assume $[x_n] = X$. For $p = 0, 1, \cdots$, let S_p be the natural partial sum projection of X onto $[x_k]_{k=0}^{np+p}$ and let $R_p = I - S_p$ be the remainder projection. By a standard renorming technique, we may assume that $\| S_p\| = \| R_p \| = 1$ for each p.

Set $Y = [f_n]$ in X^*. Now for each fixed j, $(\sum_{i=j}^{k} \hat{x}_{n_i+j})_{k=j}^{\infty}$ weak*-converges in Y^* to an element ϕ_j with $\|\phi_j\| \leqq \lambda$. (Here $\hat{x}_n$ denotes the element of Y^* defined by $\hat{x}_n(y) = y(x_n)$. Since $\| S_p \| = 1$ and $S_p x \to x \ \forall x \in X$, $\wedge$ is an isometry.) Let $\tilde{R}_p = (R_p^*|_Y)^*$ $(p = 0, 1, \cdots)$; $\tilde{R}_p$ is just the remainder projection on Y^* defined by

$$\tilde{R}_p y^* = (w^*) \sum_{i=np+p+1}^{\infty} y^*(f_i)\hat{x}_i = y^* - \sum_{i=0}^{np+p} y^*(f_i)\,\hat{x}_i.$$

On the linear span $\mathrm{sp}(\phi_n)$ of (ϕ_n), we define new norms (ρ_n) by $\rho_n(\phi) = \| \tilde{R}_n\phi \|$. Certainly $\rho_n(\phi) = \| \tilde{R}_n\phi \| \leqq \| R_n \| \| \phi \| = \|\phi \|$, so (ρ_n) is equicontinuous. Thus, there is a subsequence (ρ_{q_i}) of (ρ_n) with $\rho_{q_i}(\phi) \to \rho(\phi)$ for each $\phi \in \mathrm{sp}\,(\phi_n)$. ρ is also a norm on $\mathrm{sp}\,(\phi_n) \leqq \lambda$ and $\rho(\phi_i - \phi_j) \geqq \liminf(\phi_i - \phi_j)f_{np+1+i} = 1$ for $i \neq j$. Therefore, by Lemma 2, there are integers $m_1 < m_2 < \cdots$ so that $(\phi_{2m_i} - \phi_{2m_i+1})$ is basic in $(\mathrm{sp}\,\phi_n, \rho)$ with basis constant $\leqq 2$. Setting $\bar{\phi}_i = \phi_{2m_i} - \phi_{2m_i+1}$, we have $\rho_{q_i} \to \rho$ pointwise and (ρ_{q_i}) is equicontinuous, hence $\rho_{q_i} \to \rho$ uniformly on compact sets; in particular, uniformly on unit balls of finite dimensional subspaces of $\mathrm{sp}(\bar{\phi}_n)$. Thus, there exists a subsequence (ρ'_{q_i}) of (ρ_{q_i}) with $|\rho'_{q_i}(\phi) - \rho(\phi)| < 2\rho(\phi)$ for $\phi \in [\bar{\phi}_j]_{j=1}^{i}$ and $i = 0,1,\cdots$. Also, we may guarantee that, for each i, q'_{i+1} is large enough so that the restriction of $\tilde{S}_{q'_{i+1}}$ to $\tilde{R}_{q'_i}[\phi_j^i]_{j=1}$ is a 2-isometry for each $i = 0, 1, \cdots$. (Here $\tilde{S}_p = (S_p^* \,|\, _Y)^*$. Note that $\| \tilde{S}_p\| \leqq 1$ and $\tilde{S}_p y^* \overset{w^*}{\to} y^*$ for $y^* \in Y^*$, so that $\| \tilde{S}_p y^*\| \to \| y^* \|$ for $y^* \in Y^*$ and thus uniformly for y^* in compact subsets of Y^*.)

For each $i = 0,1,\cdots$ and $j = 0, 1, \cdots, i$ define z_{n_i+j} in X by $\hat{z}_{n_i+j} = \tilde{S}_{q'_{i+1}}\tilde{R}_{q'_i}\bar{\phi}_j$. For each i, the map $z_{n_i+j} \to \bar{\phi}_j$ extends to a 4-isometry from $[z_{n_i+j}]_{j=0}^{i}$ onto $[\bar{\phi}_j]_{j=0}^{i}$ when the latter space is given the ρ norm. Hence $(z_{n_i+j})_{j=0}^{i}$ has basis constant $\leqq 8$. But $(z_{n_i+j})_{j=0}^{i} \subset (S_{q'_{i+1}} - S_{q'_i})X$, so (z_n) is basic with basis constant $\leqq 8$. Finally, note that (z_{n_i+j}) is bounded away from 0 (since $(\bar{\phi}_j)$ is), and for each fixed j, we have $\sum_{i=j+1}^{p} \hat{z}_{n_i+j} = \tilde{S}_{q'_p}\tilde{R}_{q'_j}\bar{\phi}_j$, so that $((\| \sum_{i=j}^{p} z_{n_i+j} \|)_{p=1}^{\infty})_{j=0}^{\infty}$ is bounded. Q.E.D.

Finally we have:

COROLLARY 2. *X is quasi-reflexive if and only if every basic sequence in X spans a quasi-reflexive subspace.*

In [5] it was shown that if X is non-quasi-reflexive, then there is an infinite dimensional subspace Y of X such that X/Y is non-quasi-reflexive. As promised in the introduction, we have the following sharpening of the result.

COROLLARY 3. *If X is non-quasi-reflexive, it contains a non-quasi-reflexive subspace Y such that X/Y is non-quasi-reflexive.*

PROOF. Let (z_n) be the basic sequence of Theorem 2. Let $Y = [z_{n_k+j} | j \text{ is even}]$. The verification is straightforward. Q.E.D.

REMARK. A stronger version of Theorem 2 can be easily proved when X has a basis and X^* has the bounded approximation property. Indeed, in this case, pick a separable subspace Z of X^* so that the natural map of X into Z^* has infinite codimensional range (this is possible when X is not quasi-reflexive). By [11, Remark 4.10], there is a basis (x_n) for X with biorthogonal functionals (x_n^*) satisfying $[x_n^*] \supset Z$. It is clear that (x_n) is not k-boundedly-complete for any k.

PROBLEM 1. Suppose X has a basis. If X is not quasi-reflexive, does X possess a basis which is not k-boundedly-complete for any k? If X^* contains an infinite codimensional norming subspace, does X have a basis which is not k-shrinking for any k?

PROBLEM 2. If X is separable and X^* contains an infinite codimensional norming subspace, then does X have a quotient which has a basis which is not k-shrinking for any k?

4. The Schäffer question

We saw in Section 2 that if X^* admits only finite codimensional norming subspaces, then a norming subspace M will norm X/Y whenever Y is M-closed. Here we show that there is always a counterexample to this statement when X^* contains a norming subspace of infinite codimension.

We begin with a known lemma (cf., e.g., [4]), but include its proof for the convenience of the reader.

LEMMA 3. *Suppose Y is a subspace of X and $R: X^* \to Y^*$ is the restriction mapping ($Rx^* = x^* |_Y$ for $x^* \in Y^*$). If N is a norming subspace of Y^*, then $R^{-1}N$ is a norming subspace of X^*. In particular, if Y^* admits a norming subspace of infinite codimension, then so does X^*.*

PROOF. Let λ be the norming constant of N. Note that $R^{-1}N$ is simply the set of all extensions of functionals in N to elements of X^*.

Suppose x is a unit vector in X. If $d(x,Y) \geqq (3\lambda)^{-1}$, then there exists $x^* \in Y^\perp \subset R^{-1}N$ such that $\|x^*\| = 1$ and $x^*(x) \geqq (3\lambda)^{-1}$. (Here $d(x,Y) = \inf\{\|x+y\| : y \in Y\}$.) On the other hand, if $d(x,Y) < (3\lambda)^{-1}$, then we can pick $y \in Y$ with $\|x - y\| < (3\lambda)^{-1}$. Thus there exists $y^* \in N$ with $\|y^*\| \leqq \lambda$ and $y^*(y) = \|y\| > 1 - (3\lambda)^{-1}$. Letting $\bar{y}^*$ be a Hahn-Banach extension of y^* to an element of X^*, we have that $\bar{y}^* \in R^{-1}N$ and $\bar{y}^*(y) - \bar{y}^*(x) \leqq \lambda(3\lambda)^{-1}$. Hence $\bar{y}^*(x) \geqq \bar{y}^*(y) - 3^{-1} \geqq 1 - (3\lambda)^{-1} - 3^{-1} \geqq 3^{-1}$, whence $R^{-1}N$ is norming over X. Q.E.D.

PROPOSITION 2. *Suppose X_1 is a subspace of X for which $Y = X/X_1$ is not quasi-reflexive. Assume that X_1^* contains a norming subspace M_1 of infinite codimension. Then there exists a norming subspace M of X^* so that $X_1^\perp \cap M$ is total over Y but not norming over Y.*

PROOF. A more precise way of phrasing the final part of the conclusion is that $(Q^*)^{-1}[M]$ is a total, non-norming subspace of Y^*, where $Q : X \to Y$ is the quotient map.

Since $\dim M_1^\perp = \infty$ ($\perp$ taken in X_1^{**}), there is a biorthogonal sequence (z_i, z_i^*) with $(z_i) \subset X_1^*$ and z_i^* unit vectors in $M_1^\perp$. By replacing M_1 with $(z_i^*)_\perp$, we may assume that $[\widetilde{z_i^*}] = M_1^\perp$.

Let M_0 be the inverse image of M_1 under the natural restriction mapping R from X^* onto X_1^*. M_0 is norming by Lemma 3. Also, one checks easily that $M_0^\perp = R^* M_1^\perp = R^*[\widetilde{z_i}] = \widetilde{[R^* z_i]}$. Set $R^* z_i = \bar{z}_i$.

Since Y is not quasi-reflexive, Y^* contains a total, non-norming subspace, say N_1 (cf. [5]). Thus by [8], $Y + N_1^\perp$ is not closed in Y^{**}; hence, there exist unit vectors $(\eta_i) \subset N_1^\perp$ with $d(\eta_i, Y) \to 0$. By replacing N_1 with $(\eta_i)_\perp$, we may assume that $[\widetilde{\eta_i}] = N_1^\perp$. Now Q^{**} is onto Y^{**} since $Q: X \to Y$ is the quotient map, so we may choose $(\bar{\eta}_i) \subset X^{**}$ with $Q^{**}\bar{\eta}_i = \eta_i$. We easily check that $Q^* N_1$ is equal $(\bar{\eta}_i) \cap X_1^\perp$. Indeed, $Q^* Y^* = X_1^\perp$, and if $f \in Q^* N_1$, say $f = Q^* g$ with $g \in N_1$, then $\bar{\eta}(f) = \bar{\eta} Q^* g = (Q^* \bar{\eta}_i) g = \eta_i(g) = 0$. On the other hand, if $f \in (\bar{\eta}_i)_\perp \cap X_1^\perp$ then $f = Q^* g$, for some $g \in Y^*$, and $\eta_i g = Q^{**}\bar{\eta}_i g = \bar{\eta}_i Q^* g = \bar{\eta}_i(f) = 0$, whence $g \in (\eta_i)_\perp = N_1$.

For $\varepsilon_i > 0$ and $\varepsilon_i \to 0$ fast enough, we extend the map $\bar{z}_i^* \to \bar{z}_i^* + \varepsilon_i \bar{\eta}_i$ to a weak* automorphism on X^{**}. To do this, pick $\bar{z}_i \in X^*$ with $R\bar{z}_i = z_i$ (so that $(\bar{z}_i; \bar{z}_i^*)$ is biorthogonal) and define $T : X^* \to X^*$ by $TX^* = \Sigma\, \varepsilon_i \bar{\eta}_i(x^*) \bar{z}_i$. If $\varepsilon_i \to 0$

fast enough, $\|T\| < 1/2\beta < 1$, where β is the norming constant of M_0. Thus $I + T$ is an automorphism on X^* and it is easy to check that $M \equiv (I+T)^{-1}M_0$ is norming. Further, $M^{\perp} = (I + T^*)M_0{}^{\perp} = (I + T^*)[\widetilde{z}^*] = \overbrace{[(I + T^*)(\bar{z}_i{}^*)]}$ $= \overbrace{[\bar{z}_i^* + \varepsilon_i\bar{\eta}_i]}$.

Observe that $(Q^*)^{-1}M$ is indeed a total, non-norming subspace of Y^*, since $M \cap X_1^{\perp} = (\bar{z}_i^* + \varepsilon_i\bar{\eta}_i)_{\perp} \cap X_1^{\perp} = (\bar{\eta}_i)_{\perp} \cap X_1^{\perp}$ (because $(\bar{z}_i^*) \subset X_1^{\perp\perp}$) $= Q^*N_1$, and hence $(Q^*)^{-1}M = N_1$. Q.E.D.

We now have the main result of this section.

THEOREM 3. *If X^* contains an infinite codimensional norming subspace, then $X^* \supset M$ norming and $X \supset Y$ M-closed such that $M \cap Y^+$ fails to norm X/Y.*

PROOF. Let Y be the subspace of X from Corollary 1 such that X/Y has a norming infinite codimensional subspace in its dual. This forces X/Y to be non-quasi-reflexive, so Proposition 2 gives the desired result. Q.E.D

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